Comparative Analysis of the Solution to Fredholm Linear Integro-Differential Equations by ADM, MADM and Series Expansion Method

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Abstract

This study presents a comparative analysis of three methods for solving Fredholm linear integro-differential equations: the Adomian Decomposition Method (ADM), the Modified Adomian Decomposition Method (MADM), and the Series Expansion Method (SEM). These techniques are employed to derive approximate analytical solutions to equations that often resist exact analytical methods. The primary focus is to evaluate the accuracy, efficiency, and convergence of each method through theoretical analysis and practical examples. Graphs and tables are provided to illustrate the performance of these methods in solving selected problems. Our findings indicate that while each method has its strengths, MADM demonstrates superior accuracy in most cases, making it a promising tool for handling complex integro-differential equations in numerical analysis

Keywords: *Fredholm integro-differential equations, Adomian Decomposition Method, Modified Adomian Decomposition Method and Series Expansion Method*

1. Introduction

Some important problems in science and engineering can usually be reduced to a system of integral and integro-differential equations (Rabbani & Zarali, 2012). An integro-differential equation (IDE) is an equation that involves a combination of differential and integral operators in a single equation. The background of the study of integro-differential equations is rooted in the broader study of differential equations and integral equations. Differential equations to Scientists and Engineers provide dynamics of mathematical models that describe natural phenomena that are abound in their fields (Nwaoburu, 2020). It describes how a quantity changes with respect to one or more independent variables while an integral equation is an equation containing an unknown function under an integral sign.

pursuing analytical solutions to integro-differential equations represents a formidable yet crucial endeavor in mathematical analysis. The Decomposition Method, devised by Adomian (1988), has been pivotal in the development of analytic solution techniques. Adomian's work laid the groundwork for subsequent modifications and applications, demonstrating its efficacy in nonlinear problems

The Modified Decomposition Method (MDM) was introduced as an enhancement of the original method to address a broader class of problems. This comprehensive study highlights the modifications and their application to singular initial value problems in second-order ordinary differential equations (Wazwaz, 2006).

Employing methods such as Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM), and Series Expansion Method (SEM), researchers aim to resolve these equations analytically and numerically. Through a systematic exploration of the method's adaptability to integro-differential contexts, the study endeavors to provide a comprehensive and efficient analytic solution framework

while both ADM and MADM share the general approach of decomposing a differential equation into simpler components, MADM is specifically tailored to enhance the method's performance when dealing with integro-differential equations. The MADM aim to address challenges associated with integral terms, making it a more robust and versatile tool for a broader range of mathematical models. On the other hand, SEM represents the solution as a series expansion involving orthogonal functions. This method relies on finding appropriate basis functions and determining expansion coefficients to approximate the solution. The study aims to comprehensively investigate and compare the performance of ADM, MADM and SEM for solving Linear Fredholm Integro-Differential Equations.

Several authors have used have Used ADM, MADM and SEM in solving integro differential equation. Some have also made comparative analysis of different methods in solving integro differential equation. For instance, Shams and Tarig (2020), utilized the Adomian decomposition Sumudu transform method combined with the Pade approximant (ADST-PA method) to obtain closed-form solutions for nonlinear integro-differential equations. Additionally, they conducted a comparative study between the ADST-PA method and three other numerical methods: the Adomian decomposition Sumudu transform method (ADSTM), the homotopy perturbation method (HPM), and the variational iteration method (VIM). Their results indicate that the ADST-PA method provides a superior approximation for a wide range of nonlinear integro-differential equations compared to the existing methods. In their report, we have reviewed several recent numerical methods for solving integro- differential equations. The numerical studies demonstrated that all the methods produced highly accurate solutions for the given equations. The ADSTM, HPM, and VIM are straightforward and user-friendly. However, they do not converge to a closed form. The ADSTM method, in particular, is based on approximating the solution function by truncating the series, leading to an inaccurate solution that significantly limits the method's applicability. Asire and Najmudd (2023), presented a comparative analysis of the Adomian Decomposition Method (ADM), the Modified Adomian Decomposition Method (MADM), and the Variational Iteration Method (VIM). The primary objective of their research was to identify the most effective method between the three methods.

They said that the Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM), and Variational Iteration Method (VIM) are efficient and effective methods for solving a wide range of problems. They said that the main advantage of these methods is that they do not require the variables to be discretized. Furthermore, these are unaffected by computation round off errors.

Additionally, they concluded that while ADM requires the evaluation of an Adomian polynomial, which primarily required time-consuming algebraic calculations, VIM requires the evaluation of a Lagrangian multiplier. Also, VIM facilitates the computational work and gives the solution rapidly if compared with ADM and MADM.

Ghorieshi, *et*. al. (2011), introduced a general framework for solving nth-order integro-differential equations using the Homotopy Analysis Method (HAM) and the Optimal Homotopy Asymptotic Method (OHAM). He concluded that the OHAM is parameter-free and often achieves better accuracy than HAM at the same level of approximation. Additionally, OHAM allows for easy adjustment and control of the convergence region. In his study, a comparison through two examples shows that both HAM and OHAM are effective and accurate for solving nth-order integro-differential equations, closely matching the exact solutions

2. Methodology

Let us consider the linear fredholmn integro differential equation

$$u^n(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (2.1)$$

With $u^m(0) = \gamma_m, 0 \leq m \leq (n-1)$ $\quad$ $that\ is\ u(0) = \gamma_1, \frac{du(0)}{dx} = \gamma_2, \frac{d^2u(0)}{dx^2} = b_3, \frac{d^3u(0)}{dx^3} = \gamma_4 \ldots\ldots\ldots\ldots\ldots.. \frac{d^{n-1}u(0)}{dx^{n-1}} = \gamma_{\mathrm{n-1}}$

In this context $\gamma_1, \gamma_2, \gamma_3, \gamma_3, \ldots\ldots\ldots\ldots.. \gamma_{\mathrm{n-1}}$ denotes real constants representing the initial condition of $u(x)$ and its derivatives at 0, while $u^n(x)$ which is equivalent to $\frac{d^n u}{dx^n}$ denotes the nth derivative of the unknown function $u(x)$ and $f(x)$ is a known function. These derivatives appear both inside and outside the integral sign. The integral function's kernel, denoted as $K(x,t)$, and the function $f(x)$ are specified as real-valued functions while u(t) represents a linear function of it.

The methods under discussion include the Adomian Decomposition Method, Modified Adomian Decomposition Method and series expansion method each contributing to the advancement of solving these types of equations. The subsequent paragraphs will elaborate on these methods in detail.

2.2 Adomian decomposition method (ADM)

ADM is an analytical technique used to solve linear and nonlinear differential and integral equations. It decomposes the solution into a series of components and solves each component iteratively. The method utilizes Adomian polynomials and operator theory to obtain analytical approximations of the solution.

It is normal to integrate both sides of equation (2.1). Suppose L^{-1} is an n – fold integration operator

$$L^{-1}(u^n(x)) = L^{-1}(f(x)) + L^{-1}(\lambda \int_a^b K(x,t)u(t))dt) \quad (2.2)$$

$$u(x) = \gamma_0 + \gamma_1 x + \frac{1}{2!}\gamma_2 x^2 + \frac{1}{3!}\gamma_3 x^3 + \cdots + \frac{1}{(n-1)!}\gamma_{n-1}x^{x-1} + L^{-1}(f(x)) + L^{-1}(\lambda(\int_a^b k(x,t)u(t)dt)) \quad (2.3)$$

Equation (2.3) can be expressed as

$$u(x) = \sum_{l=0}^{n-1}\frac{1}{l!}\gamma_l x^l + L^{-1}(f(x)) + L^{-1}(\lambda \int_a^b K(x,t)u(t))dt) \quad (2.4)$$

Without loss of generality, if

$$K(x,t) = q(x)w(t) \quad (2.5)$$

Equation (2.5) implies that the kernel is separable,

Equation 2.4 can be expressed as

$$u(x) = \sum_{l=0}^{n-1}\frac{1}{l!}\gamma_l x^l + h(x) + L^{-1}q(x)(\lambda \int_a^b w(t)u(t))dt) \quad (2.6)$$

Let us rewrite (2.6) as

$$u(x) = \sum_{l=0}^{n-1}\frac{1}{l!}\gamma_l x^l + h(x) + z(x)(\lambda \int_a^b w(t)u(t))dt) \quad (2.7)$$

Were

$$h(x) = L^{-1}(f(x)) \quad (2.8)$$

$$z(x) = L^{-1}q(x) \quad (2.9)$$

$\sum_{l=0}^{n-1}\frac{1}{l!}\gamma_l x^l$ is gotten from the n-fold integrator operation

The Adomian decomposition method defines the series solution $u(x)$ by decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components. This decomposition is represented by the Adomian decomposition series. The approach involves breaking down the unknown function into its constituent components using the Adomian method, resulting in a series representation for the solution $u(x)$

Let $u(x)$ be defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (2.10)$$

Equation (2.10) is equivalent to

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) \ldots \quad (2.11)$$

The components, denoted as $u_n(x)$ for n≥0, that is $u_{1,}u_2\ u_3,\ldots\ldots\ldots\ldots\ldots\ldots$ are determined in a recursive manner within the framework of the Adomian decomposition method. This method focuses on the identification of these components, and the process involves substituting the linear Fredholm integro-differential equation to derive the solution.

If we substitute equation (2.10) into equation (2.7), we get

$$\sum_{n=0}^{\infty} u_n(x) = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + h(x) + z(x)(\lambda \int_a^b w(t) \sum_{n=0}^{\infty} u_n(t))dt) \quad (3.12)$$

Equation (2.12) can also be written as

$$u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + h(x) + z(x)(\lambda \int_a^b w(t)(u_0(t) + u_1(t) + u_2(t) + u_3(t) + u_4(t) + \cdots)dt) \quad (2.13)$$

$$\begin{aligned} u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots \\ = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + h(x) + z(x)(\lambda \int_a^b w(t)(u_0(t)) + z(x)(\lambda \int_a^b w(t)u_1(t) \\ + z(x)(\lambda \int_a^b w(t)u_2(t) + z(x)(\lambda \int_a^b w(t)u_3(t) + z(x)(\lambda \int_a^b w(t)u_4(t) \\ + \cdots)dt) \end{aligned} \quad (2.14)$$

Every term outside $z(x)(\lambda \int_a^b w(t) \sum_{n=0}^{\infty} u_n(t))dt)$ is referred to as $u_0(x)$.

Comparing both sides of equation 14

$$u_0(x) = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + h(x) \quad (2.15)$$

$$u_1(x) = z(x)\lambda \int_a^b w(t)u_0(t)dt \quad (2.16)$$

$$u_2(x) = z(x)\lambda \int_a^b w(t)u_1(t)dt \quad (2.17)$$

$$u_3(x) = z(x)\lambda \int_a^b w(t)u_2(t)dt) \quad (2.18)$$

$$u_4(x) = z(x)\lambda \int_a^b w(t)u_3(t)dt \quad (2.19)$$

$$u_5(x) = z(x)\lambda \int_a^b w(t)u_4(t)dt \quad (2.20)$$

And so on

Equation 2.15 -2.20 can be written as

$$u_{n+1}(x) = \mathrm{z(x)}\lambda \int_a^b w(t)u_n(t)dt \qquad n \geq 1 \tag{2.21}$$

From equation (2.21),

$$\left.\begin{array}{l}
u_1^{(n_1)}(x) = h_1(x) + z_1(x)(\int_{p_1}^{m_1} \left(t, u_1(t), u_2(t), \dots, u_p(t)\right) dt \\
u_2^{(n_2)}(x) = h_2(x) + z_2(x)(\int_{p_2}^{m_2} \left(t, u_1(t), u_2(t), \dots, u_p(t)\right) dt \\
u_p^{(n_3)}(x) = h_3(x) + z_3(x)(\int_{p_3}^{m_3} \left(t, u_1(t), u_2(t), \dots, u_p(t)\right) dt \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
u_p^{(n_p)}(x) = h_p(x) + z_p \int_{p_w}^{m_p} k_p\left(t, u_1(t), u_2(t), \dots, u_p(t)\right)dt \\
\text{With initial conditions:} \\
u_i^{(j)}(x_0) = u_{ij}, i = 1,2,3 \dots, w, j = 0,1,2,3 \dots, n_{i-1}
\end{array}\right\} \tag{2.22}$$

The Adomain series of $u_i(x)$can be written as the following

$$\begin{cases}
u_1(x) = \sum_{j=0}^{\infty} u_{1j} \\
u_2(x) = \sum_{j=0}^{\infty} u_{2j} \\
u_3(x) = \sum_{j=0}^{\infty} u_{3j} \\
\quad\quad \cdot \\
\quad\quad \cdot \\
\quad\quad \cdot \\
\quad\quad \cdot \\
\quad\quad \cdot \\
u_p(x) = \sum_{j=0}^{\infty} u_{pj}
\end{cases} \tag{2.23}$$

In general, $u_i(x) = \sum_{j=0}^{\infty} u_{ij}, i = 1,2,3, \dots \dots and\ j = 0,1,2, \dots \dots$ (2.24)

2.3 Modified Adomain Decomposition Method

The Modified Decomposition Method is an advancement of the Adomain Decomposition Method yields the exact solution by computing only two terms from the decomposition series, offering the advantage of computational efficiency.

From equation (2.10)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

These relationships simplify the iterative process of determining the components.

In the conventional decomposition method, the initial components, denoted as $u_0(x)$, are typically identified as the function $L^{-1}(f(x)) + \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l$ which is equivalent to $h(x) + \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l$. However, in the modified decomposition approach, as outlined in Equation (2.1), the data function $f(x)$ can be intricately divided into two distinct components. That is

$$f(x) = f_0(x) + f_1(x) \tag{2.25}$$

Consequently, the recursive relations for the modified decomposition method can be expressed as follow

$$u_0(x) = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + L^{-1}(h_0(x)) \tag{2.26}$$

$$u_1(x) = L^{-1}\big(f_1(x)\big) + z(x) \int_a^b h(t) u_o(t) dt \tag{2.27}$$

$$u_{n+1}(x) = z(x) \int_a^b h(t) u_n(t) dt \,, n \geq 1 \tag{2.28}$$

Clearly, equation (2.26 – 2.28) above exhibits reliability by enhancing solution convergence and diminishing the computational workload in comparison to the Adomian Decomposition Method.

It facilitates a faster convergence of the solution. The modified decomposition method tailors the recursive relations to enhance the convergence behavior, ensuring a more efficient and rapid attainment of the solution.

2.4 Series Expansion Method

The Taylor series method represents the solution as a power series expansion. It involves expanding the unknown function and the kernel function in Taylor series about a given point and substituting these expansions into the integral equation. By equating coefficients of like powers of x, one can obtain a sequence of equations for the coefficients of the series expansion, which can then be solved to approximate the solution

The fundamental principle behind the Series Solution Method primarily originates from utilizing Taylor series expansions of analytical functions. It's important to emphasize that for Taylor series to be applicable, the existence of derivatives of all orders is essential, prompting us to compute these derivatives accordingly. Furthermore, it's noteworthy that Taylor series centered at any point b within its domain converges to $f(x)$ within a neighborhood surrounding b

$$u(x) = \sum_{n=0}^{\infty} \frac{u^n(a)}{n!}(x-b)^n \tag{2.29}$$

When x = 0, equation (3.29) is reduced to

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2.30}$$

Or $u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$ (2.31)

From (3.6)

$$u(x) = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + h(x) + L^{-1}q(x)(\lambda \int_a^b w(t)u(t))dt)$$

Substituting equation (3.30) into equation (3.6), we get

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{l=0}^{n-1} \frac{1}{l!}\gamma_l x^l + h(x) + L^{-1}q(x)(\lambda \int_a^b w(t) \sum_{n=0}^{\infty} a_n x^n)dt\ ) \tag{2.31}$$

If $h(x)$ and $L^{-1}q(x)$ comprises elementary functions like exponential functions, trigonometric functions, etc., we should employ Taylor expansions for the functions contributing to the function. Now, equating coefficients of like powers of x on both sides, we obtain a system of equations for the coefficients a_n. Solving this system will give us the coefficients and hence the Taylor series solution for equation.

2.5 Solved Examples

Example 3.5.1: Consider the linear Fredholm integro-differential equation: $u'(x) = e^x - x + xe^x + \int_0^1 xu(t)dt$, with the initial condition $u(0) = 0$, and the exact solution is

$u(x) = xe^x$.(Asiya & Najmuddin, 2023).

Using inverse operator $L^{-1} = \int(.)dx$

We get $L^{-1}(u'(x)) = L^{-1}(e^x) - L^{-1}(x) + L^{-1}(xe^x) + L^{-1}(\mathrm{x}\int_0^1 u(t)dt)$

$$\int u'(x)dx = \int e^x dx + \int xe^x dx + \int_0^1 u(t)dt)$$

$$u(x) = e^x - \frac{x^2}{2} + xe^x - e^x + \frac{x^2}{2}\int_0^1 u(t)dt + c$$

Where c is the constant of integration. Using the initial condition $u(0) = 0$

$$c = 0$$

We get

$u(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u(t)dt.$

Adomain Decomposition Method

From (3.10) $u(x) = \sum_{n=0}^{\infty} u_n(x)$

Substituting equation (2.10) into (i)

$$\sum_{n=0}^{\infty} u_n(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 \sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) \ldots = xe^x - \frac{x^2}{2} + \frac{x^2}{2} + \int_0^1 (u_0(t) + u_1(t) + u_2(t) + u_3(t) + u_4(t) \ldots\ldots)dt$$

Clearly $u_0(x) = xe^x - \frac{x^2}{2}$

$$u_{n+1}(x) = \frac{x^2}{2}\int_0^1 \sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_1(x) = \frac{x^2}{2}\int_0^1 u_0(t)dt = \frac{x^2}{2}\int_0^1 (te^t - \frac{t^2}{2})dt =$$

$$\frac{x^2}{2}(-e^t + te^t - \frac{t^3}{6})\begin{matrix}1\\0\end{matrix}$$

$$u_1(x) = \frac{5}{12}x^2$$

Similarly, $u_2(x) = \frac{x^2}{2}\int_0^1 \left(\frac{5t^2}{12}\right)dt = \frac{5x^2}{72}$

$$u_3(x) = \frac{x^2}{2}\int_0^1 \left(\frac{5t^2}{72}\right)dt = \frac{5x^2}{432}$$

And so on

$$u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots = xe^x - \frac{1}{2}x^2 + \frac{5}{12}x^2 + \frac{5x^2}{72} + \frac{5x^2}{432} + \cdots ..$$

Hence, $u(x) \approx xe^x + \frac{431}{432}x^2$

Modified Adomain Decomposition Method

Given that $u(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u(t)dt$. We will need to split $f_0(x) = xe^x$ and

$f_1(x) = -\frac{x^2}{2}$

$$u_0(x) = f_0(x) = xe^x$$

$$u_1(x) = -\frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u(t)dt =$$

$$u_1(x) = -\frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u_0(x)dt =$$

$$u_1(x) = -\frac{x^2}{2} + \frac{x^2}{2}\int_0^1 xe^x dt =$$

$$u_1(x) = -\frac{x^2}{2} + \frac{x^2}{2} = 0 \text{ where } \int_0^1 xe^x dt = 1$$

Also, $u_2(x) = 0$ and so on. This implies th at $u_{n+1}(x) = 0$ for $n \geq 1$

Hence, the solution becomes $u(x) = xe^x$.

Series Expansion Method

From equation (i) $u(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u(t)dt.$

Let $u(x) = \sum_{n=0}^{\infty} a_n x^n$

$u(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u(t)dt$. Can be written as

$\sum_{n=0}^{\infty} a_n x^n = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 \sum_{n=0}^{\infty} a_n t^n \, dt$

Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

On substitution, $\sum_{n=0}^{\infty} a_n x^n = x\sum_{n=0}^{\infty} \frac{x^n}{n!} - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 \sum_{n=0}^{\infty} a_n t^n \, dt$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - \frac{x^2}{2} + \frac{x^2}{2} \int_0^1 \sum_{n=0}^{\infty} a_n t^n \, dt$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - \frac{x^2}{2} + \frac{x^2}{2} \left[\sum_{n=0}^{\infty} a_n \frac{t^{n+1}}{n+1} \right|_0^1$$

$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} - \frac{x^2}{2} + \frac{x^2}{2} \sum_{n=0}^{\infty} \frac{a_n}{n+1}$ (ii)

Equation (ii) can be written as

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} \ldots \right) - \frac{x^2}{2} + \frac{x^2}{2} (a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots)$$

Comparing co-efficient

$$a_0 = 0$$

$$a_1 = 1$$

$$a_3 = \frac{1}{2!} = \frac{1}{2}$$

$$a_4 = \frac{1}{3!} = \frac{1}{6}$$

$$a_5 = \frac{1}{4!} = \frac{1}{24}$$

Let's calculate an approximate value for a_2

$$a_2 \approx 1 - \frac{1}{2} + \frac{1}{2} (a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} + \frac{a_4}{5} \ldots)$$

$$a_2 \approx 1 - \frac{1}{2} + \frac{1}{2} (0 + \frac{1}{2} + \frac{a_2}{3} + \frac{\left(\frac{1}{2}\right)}{4} + \frac{\left(\frac{1}{6}\right)}{5} + \frac{\left(\frac{1}{24}\right)}{6} \ldots)$$

$$a_2 \approx 1 - \frac{1}{2} + \frac{1}{2} (0 + \frac{1}{2} + \frac{a_2}{3} + \frac{1}{8} + \frac{1}{30} + \frac{1}{144} \ldots)$$

$$a_2 - \frac{a_2}{6} \approx 1 - \frac{1}{2} + \frac{1}{2} (0 + \frac{1}{2} + \frac{1}{8} + \frac{1}{30} + \frac{1}{144} \ldots)$$

$$a_2 - \frac{a_2}{3} \approx 0.8326$$

On evaluation, $a_2 \approx 0.999$

Hence the series becomes $u(x) \approx x + 0.999\frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} \ldots$

Example 3.5.2: Let us Consider the system of Fredholm integro-differential equations:

$$\begin{cases} M''(x) = \frac{3x}{10} + 6 - \int_0^1 2xt(M(t) - 3N(t))dt \\ N''(x) = 15x + \frac{4}{5} - \int_0^1 3(2x + t^2)(M(t) - 2N(t))dt \end{cases}$$

with the initial conditions $M(0) = 1, N(0) = -1, M'(0) = 0$ and $N'(0) = 2$ and the exact solutions are $M(x) = 3x^2 + 1$ and $N(x) = x^3 + 2x - 1$ (Asiya & Najmuddin, 2023).

Taking inverse operator $L^{-1} = \iint(.)dxdx$ with the initial conditions $M(0) = 1, N(0) = -1, M'(0) = 0$ and $N'(0) = 2$

We get

$$L^{-1}\big(M''(x)\big) = L^{-1}\Big(\frac{3x}{10} + 6 - \int_0^1 2xt\big(M(t) - 3N(t)\big)dt\Big)$$

$$L^{-1}(N''(x)) = L^{-1}(15x + \frac{4}{5} - \int_0^1 3(2x + t^2)(M(t) - 2N(t))dt)$$

$$\begin{cases} \mathrm{M}(x) = 1 + \frac{1}{20}x^3 + 3x^2 - \frac{1}{3}x^3 \int_0^1 t(G(t) - 3N(t))dt \\ N(x) = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3 \int_0^1 (G(t) - 2H(t))dt - \frac{3}{2}x^2 \int_0^1 t^2(G(t) - 2H(t))dt \end{cases} \quad (iii)$$

Here $m(x) = 1 + \frac{1}{20}x^3 + 3x^2$

And $n(x) = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2$

Adomain Decomposition Method

$$\begin{cases} M(x) = 1 + \frac{1}{20}x^3 + 3x^2 - \frac{1}{3}x^3 \int_0^1 t(M(t) - 3N(t))dt \\ M(x) = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3 \int_0^1 (N(t) - 2H(t))dt - \frac{3}{2}x^2 \int_0^1 t^2(M(t) - 2N(t))dt \end{cases} \quad (iii)$$

From equation 10, $M(x) = \sum_{n=1}^{\infty} N_n(x)$ $and\ N(x) = \sum_{n=1}^{\infty} N_n(x)$

Equation iii can be written as

$$\begin{cases} \sum_{n=0}^{\infty} M_n(x) = 1 + \frac{1}{20}x^3 + 3x^2 - \frac{1}{3}x^3 \int_0^1 \sum_{n=0}^{\infty} t(M_n(t) - 3N_n(t))dt \\ \sum_{n=0}^{\infty} N_n(x) = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3 \int_0^1 \sum_{n=0}^{\infty} (M_n(x) - 2N_n)dt \end{cases}$$

$$-\frac{3}{2}x^2 \int_0^1 \sum_{n=0}^{\infty} t^2(M_n(t)\, t^2(M(t) - 2N_n(\mathrm{t}))dt$$

Clearly

$$M_0(x) = m(x)$$

$$N_0(x) = n(x)$$

This implies that

$$M_0(x) = 1 + \frac{1}{20}x^3 + 3x^2$$

$$N_0(x) = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2$$

So,

$$\begin{cases} M_{n+1}(x) = -\frac{1}{3}x^3 \int_0^1 \sum_{n=0}^{\infty} t(M_n(t) - 3N_n(t))dt \\ N_{n+1}(x) = x^3 \int_0^1 \sum_{n=0}^{\infty} t^2(M_n(x) - 2N_n)dt \end{cases}$$

$$M_1(x) = -\frac{1}{3}x^3 \int_0^1 t(\, M_0(t) - 3N_0(t))dt$$

$$M_1(x) = -\frac{1}{3}x^3 \int_0^1 t(1 + \frac{1}{20}t^3 + 3t^2) - 3\left(-1 + 2t + \frac{5}{2}t^3 + \frac{2}{5}t^2\right) dx$$

$$M_1(x) = 0.34667x^3$$

Similarly

$$N_1(x) = x^3 \int_0^1 (M_0(t) - 2N_0(\mathrm{t}))dt - \frac{3}{2}x^2 \int_0^1 t^2(M_0(\mathrm{t}) - 2N_n(\mathrm{t}))dt$$

$$N_1(x) = x^3 \int_0^1 (\left(1 + \frac{1}{20}x^3 + 3x^2\right) - 2\left(-1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2\right))dt - \frac{3}{2}x^2 \int_0^1 (t^2(1 + \frac{1}{20}x^3 + 3x^2) - 2(-1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2)(\mathrm{t}))dt$$

$$N_1(x) = 0.6375x^2 - 0.49583x^3$$

And so, on

Hence $M(x) = \sum_{n=0}^{\infty} M_n(x) = M_0(x) + M_1(x) + M_2(x) + \cdots$

Will be $M(x) \approx 1 + \frac{1}{20}x^3 + 3x^2 + 0.34667x^3$ *and* $N(x) = \sum_{n=0}^{\infty} N_n(x)$

$$N(x) \approx -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 + 0.6375x^2 - 0.49583x^3$$

Which is reduced to

$$M(x) \approx 1 + 3x^2 + 0.39667x^3$$

$$N(x) \approx -1 + 2x + 1.0375x^2 + 2.0042x^3$$

Modified Adomain Decomposition Method

From equation (iii), splitting $m(x)$ into two parts i.e., $m_0(x) = 3x^2 + 1, m_1(x) = \frac{1}{20}x^3$. Also, splitting $n(x)$ into two parts i.e., $n_0(x) = -1 + 2x + x^3, n_1(x) = \frac{2}{5}x^2 + \frac{3}{2}x^3$ and use recursive relations to obtain

and,

$$\begin{cases} M_0(x) = 3x^2 + 1 \\ N_0(x) = x^3 + 2x - 1. \end{cases}$$

$$\begin{cases} M_1(x) = \frac{1}{20}x^3 - \frac{1}{3}x^3\int_0^1 t(M_0(t) - 3N_0(t))dt \\ N_1(x) = \frac{3}{2}x^3 + \frac{2}{5}x^2 - x^3\int_0^1 (M_0(t) - 2N_0(t))dt - \frac{3}{2}x^2\int_0^1 t^2(M_0(t) - 2N_0(t))dt \\ \end{cases}$$

$$M_1(x) = \frac{1}{20}x^3 - \frac{1}{3}x^3\int_0^1 t\big(M_0(t) - 3N_0(t)\big)dt$$

$$= \frac{1}{20}x^3 - \frac{1}{3}x^3\int_0^1 t((3t^2+1) - 3(t^3+2t-1)dt = \frac{1}{20}x^3 - \frac{1}{3}x^3\left(\frac{3}{20}\right) = 0$$

$$N_1(x) = \frac{3}{2}x^3 + \frac{2}{5}x^2 - x^3\int_0^1 \big(M_0(t) - 2N_0(t)\big)dt - \frac{3}{2}x^2\int_0^1 t^2\big(M_0(t) - 2N_0(t)\big)dt$$

$$= \frac{3}{2}x^3 + \frac{2}{5}x^2 - x^3\int_0^1 ((3t^2+1) - 2\,(t^3+2t-1))\mathrm{dt} - \frac{3}{2}x^2\int_0^1 t^2((3x^2+1)$$

$$- 2((t^3+2t-1)\mathrm{dt} = \frac{3}{2}x^3 + \frac{2}{5}x^2 - \frac{3}{2}x^3 - \frac{3}{2}x^2\left(\frac{4}{15}\right) = 0.$$

$$M_1(x) = 0 \; and \; N_1(x) = 0$$

Clearly

$$\begin{cases} M_{\mathrm{n+1}}(x) = 0 \\ N_{\mathrm{n+1}}(x) = 0 \quad \text{for } n \geq 1 \\ \end{cases}$$

Hence,

$$M(x) = 3x^2 + 1$$

$$N(x) = x^3 + 2x - 1$$

Which is equivalent to the exact solution

Series Expansion Method

From equation (iii),

$$\begin{cases} M(x) = 1 + \frac{1}{20}x^3 + 3x^2 - \frac{1}{3}x^3 \int_0^1 t(M(t) - 3N(t))dt \\ N(x) = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3 \int_0^1 (M(t) - 2N(t))dt - \frac{3}{2}x^2 \int_0^1 t^2(M(t) - 2N(t))dt \end{cases}$$

Let $M(x) = \sum_{n=0}^{\infty} a_n x^n$ and $N(x) = \sum_{n=0}^{\infty} b_n x^n$

Equation (iii) can be written as

$$\begin{cases} \sum_{n=0}^{\infty} a_n x^n = 1 + \frac{1}{20}x^3 + 3x^2 - \frac{1}{3}x^3 \int_0^1 t(\sum_{n=0}^{\infty} a_n t^n - 3\sum_{n=0}^{\infty} b_n t^n)dt \\ \sum_{n=0}^{\infty} b_n x^n = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3 \int_0^1 \left(\sum_{n=0}^{\infty} a_n t^n - 2\sum_{n=0}^{\infty} b_n x^n\right)dt \end{cases}$$

$$-\frac{3}{2}x^2 \int_0^1 t^2 \left(\sum_{n=0}^{\infty} a_n t^n - 2\sum_{n=0}^{\infty} b_n t^n\right)dt$$

From

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \frac{1}{20}x^3 + 3x^2 - \frac{1}{3}x^3 \int_0^1 t\left(\sum_{n=0}^{\infty} a_n t^n - 3\sum_{n=0}^{\infty} b_n t^n\right)dt$$

We have

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \frac{1}{20}x^3 + 3x^2 - \frac{1}{3}x^3 \int_0^1 \left(\sum_{n=0}^{\infty} a_n t^{n+1} - 3\sum_{n=0}^{\infty} b_n t^{n+1}\right)dt$$

$$\int_0^1 \left(\sum_{n=0}^{\infty} a_n t^{n+1} - 3\sum_{n=0}^{\infty} b_n t^{n+1}\right)dt = \sum_{n=0}^{\infty} \left(\frac{a_n}{n+2} - \frac{b_n}{n+2}\right)$$

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \frac{1}{20}x^3 + 3x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^3 \sum_{n=0}^{\infty} \left(\frac{a_n}{n+2} - \frac{b_n}{n+2}\right)$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + \frac{1}{20}x^3 + 3x^2 - \frac{1}{3}x^3 \left(\left(\frac{a_0}{2} + \frac{a_1}{3} \dots\right) + \left(\frac{b_0}{2} + \frac{b_1}{3} \dots\right)\right)$$

$$a_0 = 1, a_1 = 0, a_2 = 3$$

$$a_4 = 0, n \geq 0$$

To get an approximate value for a_3, we need to compute at least $b_0, b_1, \dots \dots$

From

$$\sum_{n=0}^{\infty} b_n x^n = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3 \int_0^1 \left(\sum_{n=0}^{\infty} a_n t^n - 2\sum_{n=0}^{\infty} b_n t^n\right) dt - \frac{3}{2}x^2 \int_0^1 t^2 \left(\sum_{n=0}^{\infty} a_n t^n - 2\sum_{n=0}^{\infty} b_n t^n\right) dt$$

$$\sum_{n=0}^{\infty} b_n x^n = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3 \int_0^1 \left(\sum_{n=0}^{\infty} a_n t^n - 2\sum_{n=0}^{\infty} b_n t^n\right) dt - \frac{3}{2}x^2 \int_0^1 \left(\sum_{n=0}^{\infty} a_n t^{n+2} - 2\sum_{n=0}^{\infty} b_n t^{n+2}\right) dt$$

$$\sum_{n=0}^{\infty} b_n x^n = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3 \left(\sum_{n=0}^{\infty} \frac{a_n}{n+1} - 2\sum_{n=0}^{\infty} \frac{b_n}{n+1}\right) - \frac{3}{2}x^2 \left(\sum_{0}^{\infty} \frac{a_n}{n+3} - 2\frac{b_n}{n+3}\right)$$

The above equation can be written as

$$b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3\left(\left(a_0 + \frac{a_1}{2} + \cdots\right) - 2\left(b_0 + \frac{b_1}{2} + \cdots\right)\right) - \frac{3}{2}x^2\left(\left(\frac{a_0}{3} + \frac{a_1}{4} + \cdots\right) - 2\left(\frac{b_0}{3} + \frac{b_1}{4} + \cdots\right)\right)$$

Comparing co efficient

$$b_0 = -1$$

$$b_1 = 2$$

$$b_4 = 0$$

$$b_n = 0, n \geq 4$$

$$b_2 \approx \frac{2}{5} - \frac{3}{2}\left[\left(\frac{1}{3}\right) - 2\left(\frac{-1}{3} + \frac{2}{4}\right)\right]$$

$$b_2 \approx 0.4$$

$$b_3 \approx \frac{5}{2} - 1\left[\left(1 - 2\left(-1 + \frac{2}{2}\right)\right)\right]$$

$$b_3 \approx 1.5$$

Similarly,

$$a_3 \approx \frac{1}{20} - \frac{1}{3}\left[\left(\frac{1}{2}\right) - 3\left(-\frac{1}{2} + \frac{2}{3}\right)\right]$$

$$a_3 \approx 0.05$$

Hence, $M(x) \approx 1 + 3x^2 + 0.05x^3$ and $N(x) \approx -1 + 2x + 0.4x^2 + 1.5x^3$

3. Results

Table 3.1: Exact and Approximate Solution by ADM, MADM and SEM For Example 1with step size 0.01

x	Exact	ADM	MADM	SEM	|Ex-ADM|	|Ex-MADM|	|Ex-SEM|
0.01	0.010100502	0.01020027	0.010100502	0.010050452	9.97685E-05	0.0000000	5.005E-05
0.02	0.020404027	0.020803101	0.020404027	0.020203827	0.000399074	0.0000000	0.0002002
0.03	0.030913636	0.031811553	0.030913636	0.030463186	0.000897917	0.0000000	0.00045045
0.04	0.041632431	0.043228727	0.041632431	0.040831631	0.001596296	0.0000000	0.0008008
0.05	0.052563555	0.055057768	0.052563555	0.051312305	0.002494213	0.0000000	0.00125125
0.06	0.063710193	0.067301859	0.063710193	0.061908392	0.003591667	0.0000000	0.0018018
0.07	0.075075573	0.07996423	0.075075573	0.072623122	0.004888657	0.0000000	0.002452451
0.08	0.086662965	0.093048151	0.086662965	0.083459763	0.006385185	0.0000000	0.003203202
0.09	0.098475686	0.106556936	0.098475686	0.094421631	0.008081250	0.0000000	0.004054054
0.10	0.110517092	0.120493944	0.110517092	0.105512083	0.009976852	0.0000000	0.005005008
0.11	0.122790588	0.134862578	0.122790588	0.116734523	0.012071991	0.0000000	0.006056065
0.12	0.135299622	0.149666289	0.135299622	0.128092397	0.014366667	0.0000000	0.007207225
0.13	0.14804769	0.164908569	0.14804769	0.139589199	0.016860880	0.0000000	0.008458491
0.14	0.161038332	0.180592961	0.161038332	0.151228468	0.019554630	0.0000000	0.009809864
0.15	0.174275136	0.196723053	0.174275136	0.163013789	0.022447917	0.0000000	0.011261347
0.16	0.187761739	0.21330248	0.187761739	0.174948796	0.025540741	0.0000000	0.012812944
0.17	0.201501825	0.230334927	0.201501825	0.187037168	0.028833102	0.0000000	0.014464657
0.18	0.215499125	0.247824125	0.215499125	0.199282633	0.032325000	0.0000000	0.016216492
0.19	0.229757424	0.265773859	0.229757424	0.211688969	0.036016435	0.0000000	0.018068455
0.20	0.244280552	0.284187959	0.244280552	0.224260000	0.039907407	0.0000000	0.020020552

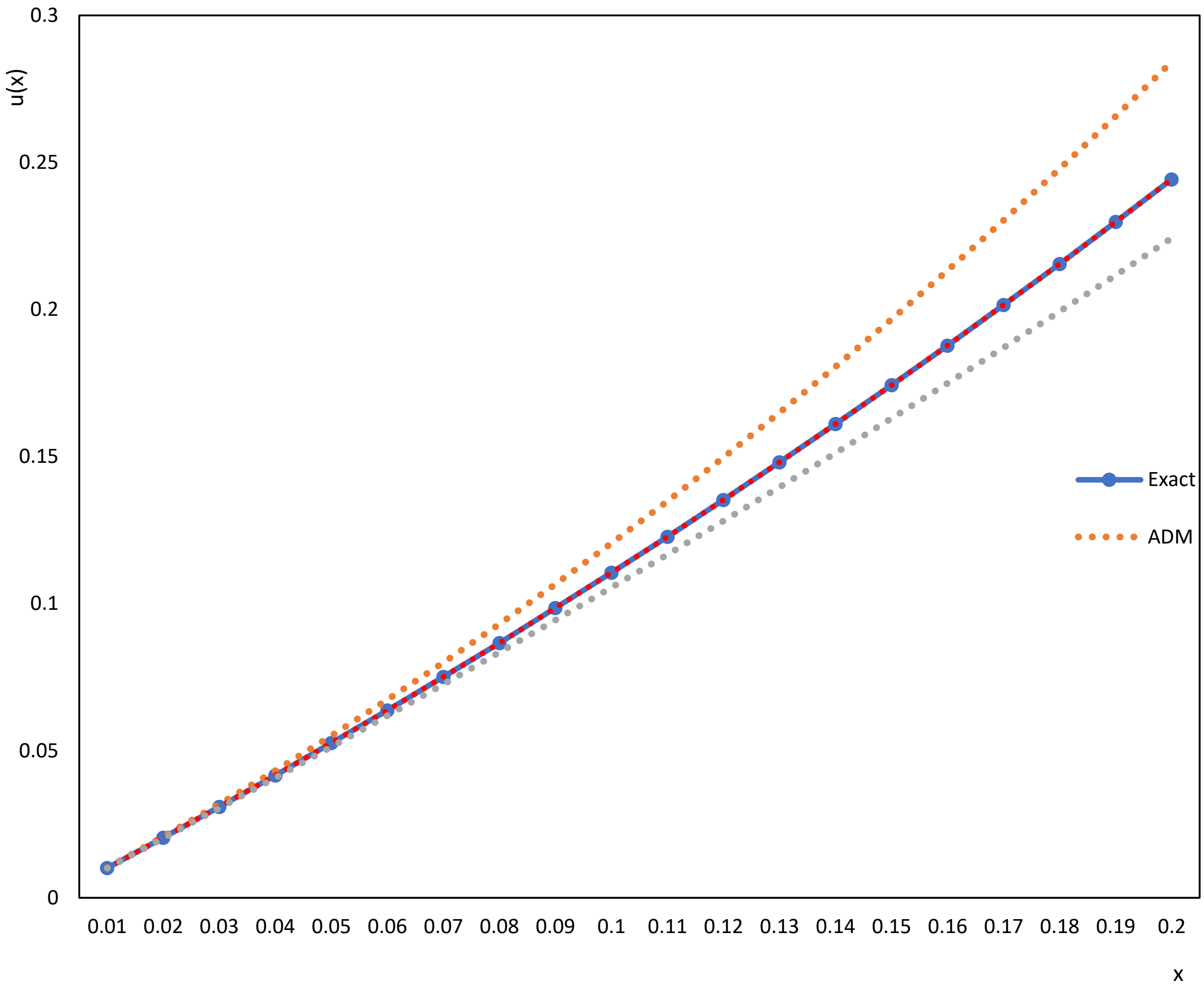

Figure 3.1. Exact and Approximate Solution by ADM, MADM and SEM For Example 1 with step size 0.01

Table 3.2. Root Mean Square Error for Example 1 with step size 0.01

Root Mean Square Error (RMSE)	
ADM	0.018964759
MADM	0
SEM	0.00951404

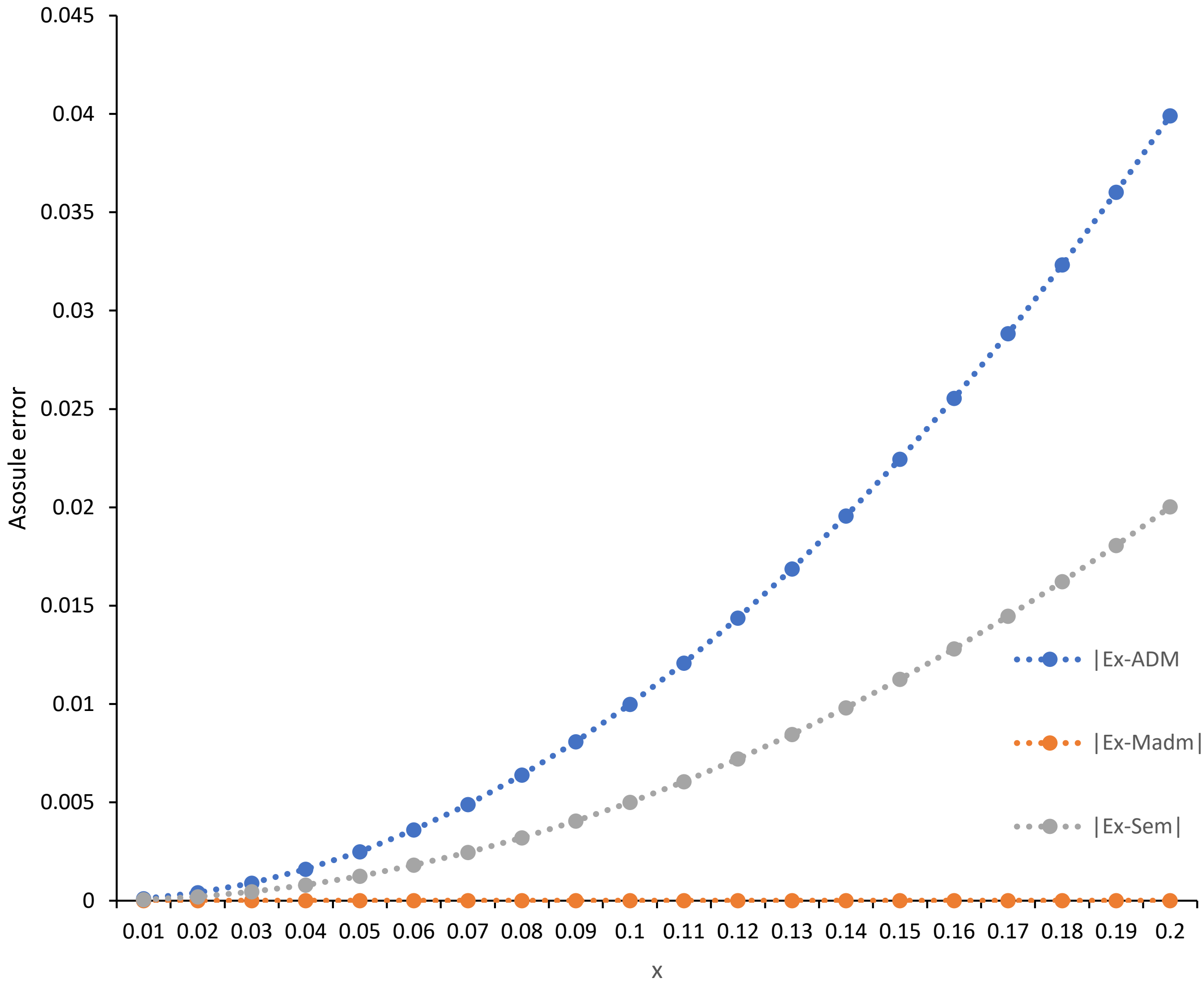

Figure 3.2. Absolute error by ADM, MADM and SEM For Example 1 with step size 0.01

Table 3.3: Exact and Approximate Solution by ADM, MADM and SEM For Example 1 with step size 0.05

X	Exact	ADM	MADM	SEM	|Ex-Adm|	|Ex-Madm|	|Ex-Sem|
0.05	0.052563555	0.05505777	0.052564	0.0513123	0.002494213	0.0000000	0.00125125
0.10	0.110517092	0.12049394	0.110517	0.1055121	0.009976852	0.0000000	0.00500501
0.15	0.174275136	0.19672305	0.174275	0.1630138	0.022447917	0.0000000	0.01126135
0.20	0.244280552	0.28418796	0.244281	0.22426	0.039907407	0.0000000	0.02002055
0.25	0.321006354	0.38336168	0.321006	0.289723	0.062355324	0.0000000	0.03128337
0.30	0.404957642	0.49474931	0.404958	0.3599063	0.089791667	0.0000000	0.04505139
0.35	0.496673642	0.61889008	0.496674	0.4353461	0.122216435	0.0000000	0.06132751
0.40	0.596729879	0.75635951	0.59673	0.5166133	0.159629630	0.0000000	0.08011655
0.45	0.705740483	0.90777173	0.70574	0.6043145	0.202031250	0.0000000	0.10142599
0.50	0.824360635	1.07378193	0.824361	0.6990938	0.249421296	0.0000000	0.12526689
0.55	0.95328916	1.25508893	0.953289	0.8016343	0.301799769	0.0000000	0.15165485
0.60	1.09327128	1.45243795	1.093271	0.91266	0.359166667	0.0000000	0.18061128
0.65	1.245101539	1.66662353	1.245102	1.0329368	0.421521991	0.0000000	0.2121647
0.70	1.409626895	1.89849264	1.409627	1.1632746	0.488865741	0.0000000	0.24635231
0.75	1.587750012	2.14894793	1.58775	1.3045283	0.561197917	0.0000000	0.28322169
0.80	1.780432743	2.41895126	1.780433	1.457600	0.638518519	0.0000000	0.32283274
0.85	1.988699824	2.70952737	1.98870	1.623440	0.720827546	0.0000000	0.36525981
0.90	2.213642800	3.0217678	2.213643	1.8030488	0.808125000	0.0000000	0.41059405
0.95	2.456424176	3.35683506	2.456424	1.9974782	0.900410880	0.0000000	0.45894601
1.00	2.718281828	3.71596701	2.718282	2.2078333	0.997685185	0.0000000	0.5104485

Table 3.4. Root Mean Square Error for Example 1 with step size 0.05

Root Mean Square Error (RMSE)	
ADM	0.47411898
MADM	0
SEM	0.240693898

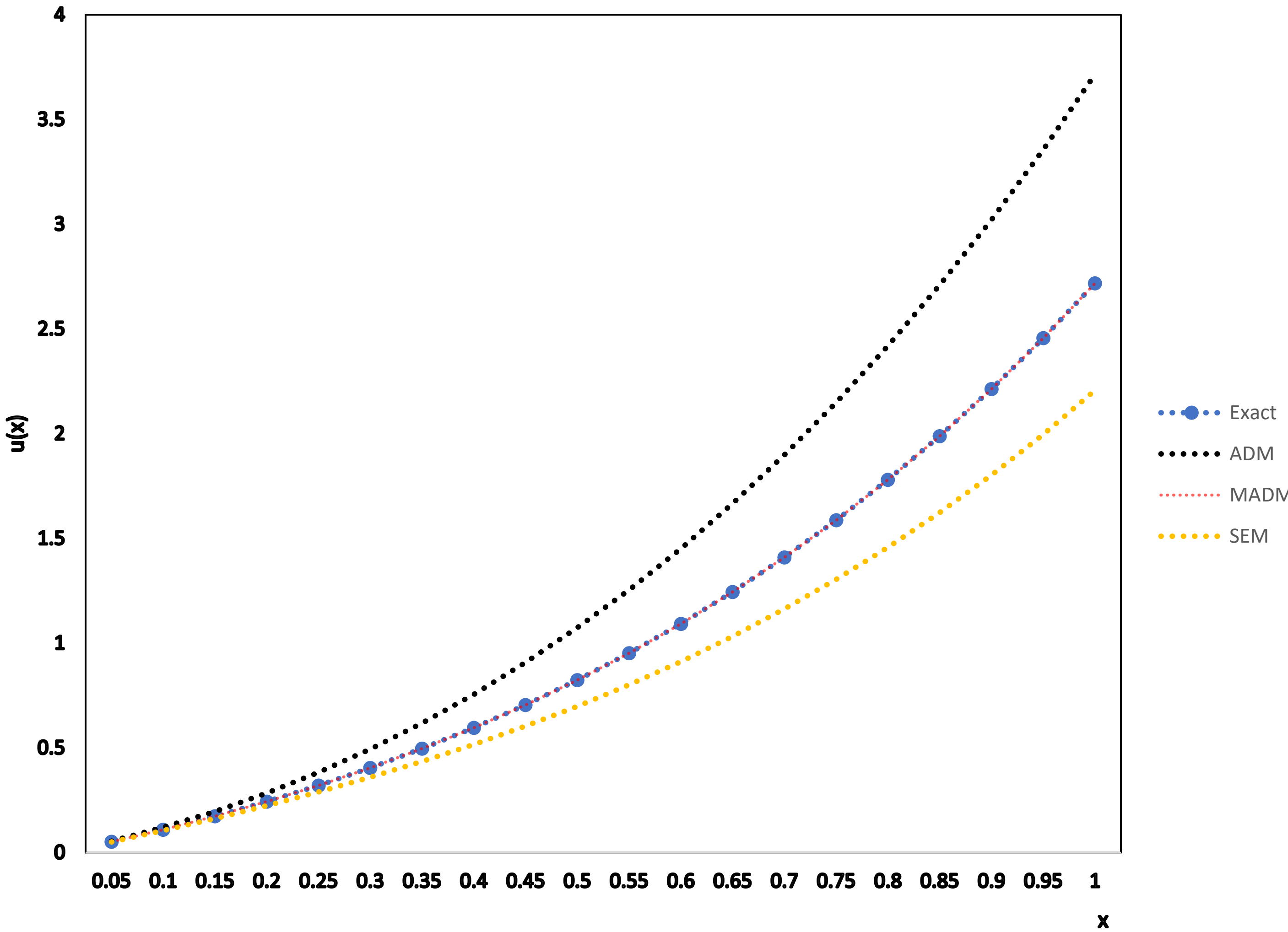

Figure 3.3: Exact and Approximate Solution by ADM, MADM and SEM For Example 1 with step size 0.05

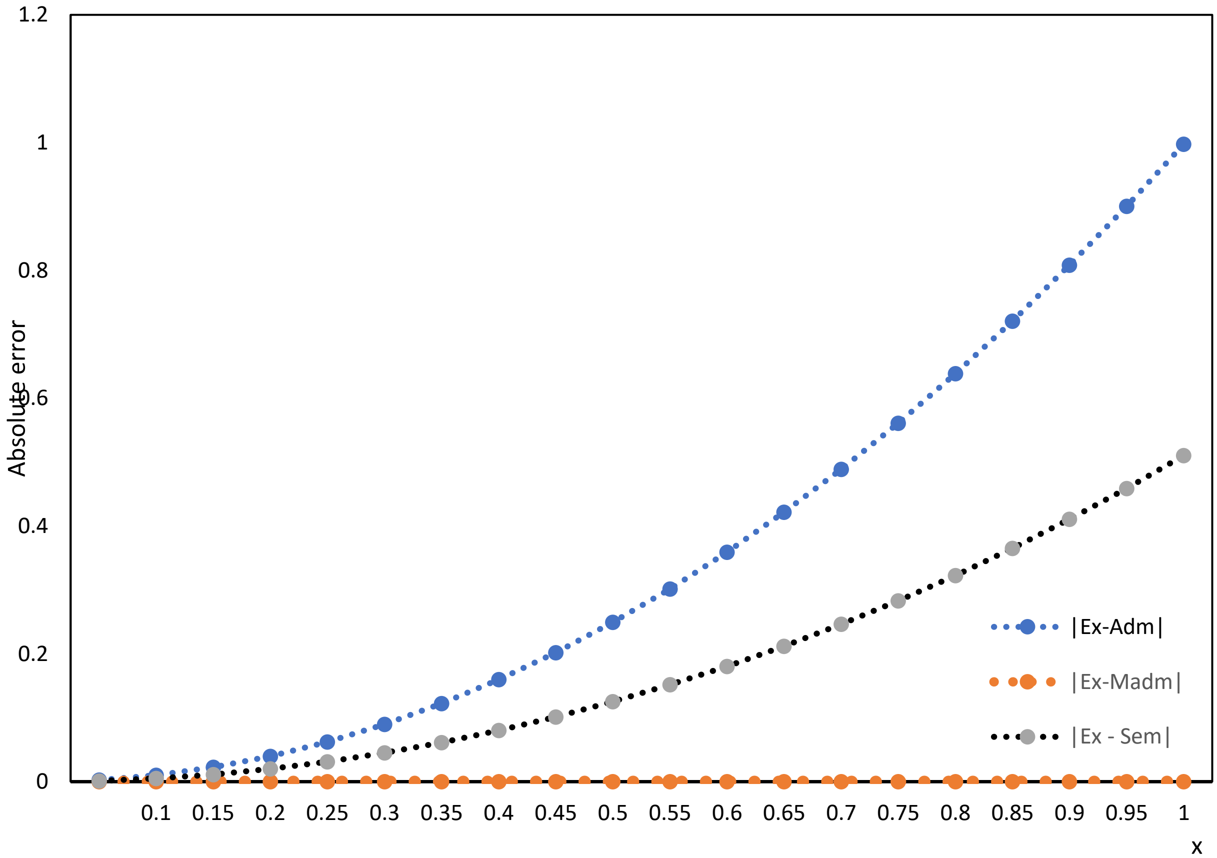

Figure 3.4: Absolute error of ADM, MADM and SEM For Example 1 with step size 0.05

Table 3.5: Exact and Approximate Solution by ADM, MADM and SEM for Example 2(M(x)) with step size 0.01

X	Exact	ADM	MADM	SEM	|Ex-Adm|	|Ex-Mad|	|Ex-Sem|
0.00	1.0000	1.00000000	1.0000	1.00000000	0.00000	0.000000	0.000000
0.01	1.0003	1.0003004	1.0003	1.00030005	3.97E-07	0.000000	5E-08
0.02	1.0012	1.00120317	1.0012	1.00120040	3.17E-06	0.000000	4E-07
0.03	1.0027	1.00271071	1.0027	1.00270135	1.07E-05	0.000000	1.35E-06
0.04	1.0048	1.00482539	1.0048	1.00480320	2.54E-05	0.000000	3.2E-06
0.05	1.0075	1.00754958	1.0075	1.00750625	4.96E-05	0.000000	6.25E-06
0.06	1.0108	1.01088568	1.0108	1.01081080	8.57E-05	0.000000	1.08E-05
0.07	1.0147	1.01483606	1.0147	1.01471715	0.000136	0.000000	1.715E-05
0.08	1.0192	1.0194031	1.0192	1.01922560	0.000203	0.000000	2.56E-05
0.09	1.0243	1.02458917	1.0243	1.02433645	0.000289	0.000000	3.645E-05
0.10	1.0300	1.03039667	1.0300	1.03005000	0.000397	0.000000	5E-05
0.11	1.0363	1.03682797	1.0363	1.03636655	0.000528	0.000000	6.655E-05
0.12	1.0432	1.04388545	1.0432	1.04328640	0.000685	0.000000	8.64E-05
0.13	1.0507	1.05157148	1.0507	1.05080985	0.000871	0.000000	0.0001099
0.14	1.0588	1.05988846	1.0588	1.05893720	0.001088	0.000000	0.0001372
0.15	1.0675	1.06883876	1.0675	1.06766875	0.001339	0.000000	0.0001688
0.16	1.0768	1.07842476	1.0768	1.07700480	0.001625	0.000000	0.0002048
0.17	1.0867	1.08864884	1.0867	1.08694565	0.001949	0.000000	0.0002457
0.18	1.0972	1.09951338	1.0972	1.09749160	0.002313	0.000000	0.0002916
0.19	1.1083	1.11102076	1.1083	1.10864295	0.002721	0.000000	0.0003430
0.20	1.1200	1.12317336	1.1200	1.12040000	0.003173	0.000000	0.0004000

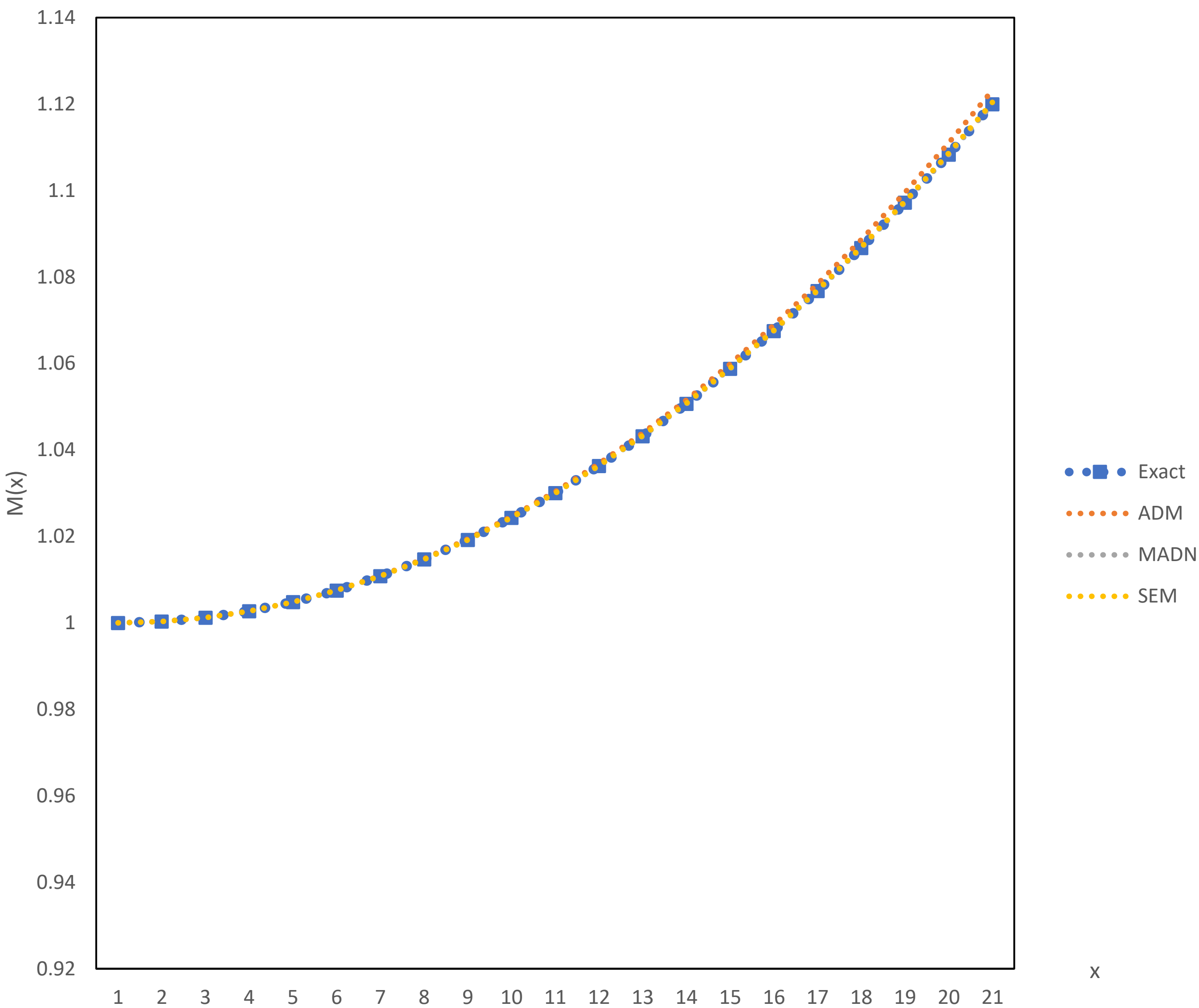

Figure 3.5: Exact and Approximate Solution by ADM, MADM And SEM for Example 2, M(x) with step size 0.01.

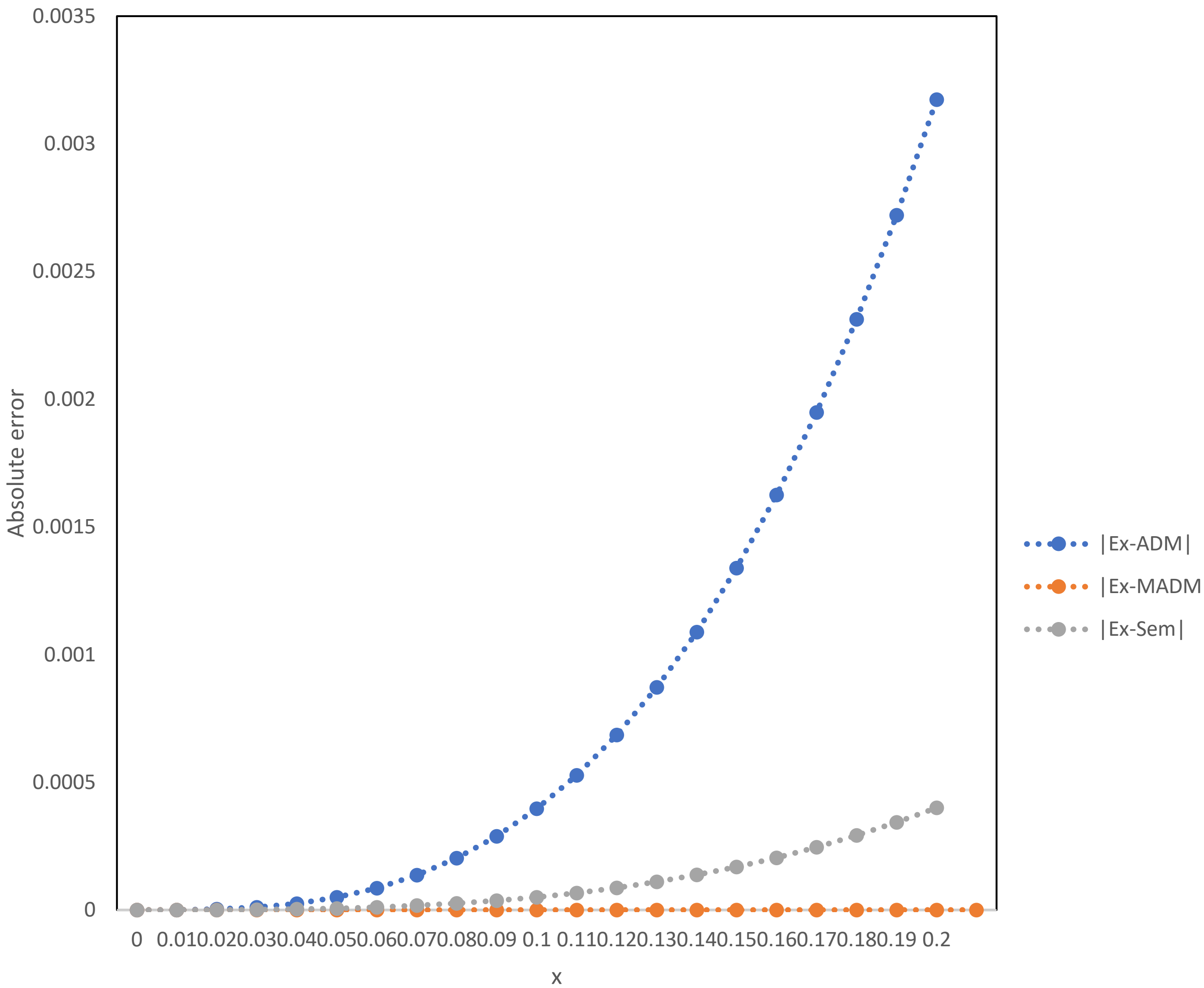

Figure 3.6: Absolute error by ADM, MADM And SEM for Example 2, M(x) with step size 0.01.

Table 3.6. Root Mean Square Error for Example 2 M(x) with step size 0.01

Root Mean Square Error (RMSE)	
ADM	0.001274
MADM	0
SEM	0.000161

Table 3.7: Exact and Approximate Solution by ADM, MADM and SEM for Example 2(M(x)) with step size 0.05

X	Exact	ADM	MADM	SEM	|Ex-Adm|	|Ex-Mad|	|Ex-Sem|
0.00	1.0000	1.000000	1.0000	1.000000	0.000000	0.0000	0.000000
0.05	1.0075	1.007550	1.0075	1.007506	4.96E-05	0.0000	6.25E-06
0.10	1.0300	1.030397	1.0300	1.030050	0.000397	0.0000	5E-05
0.15	1.0675	1.068839	1.0675	1.067669	0.001339	0.0000	0.000169
0.20	1.1200	1.123173	1.1200	1.120400	0.003173	0.0000	0.000400
0.25	1.1875	1.193698	1.1875	1.188281	0.006198	0.0000	0.000781
0.30	1.2700	1.280710	1.2700	1.271350	0.01071	0.0000	0.001350
0.35	1.3675	1.384507	1.3675	1.369644	0.017007	0.0000	0.002144
0.40	1.4800	1.505387	1.4800	1.483200	0.025387	0.0000	0.003200
0.45	1.6075	1.643647	1.6075	1.612056	0.036147	0.0000	0.004556
0.50	1.7500	1.799584	1.7500	1.756250	0.049584	0.0000	0.006250
0.55	1.9075	1.973496	1.9075	1.915819	0.065996	0.0000	0.008319
0.60	2.0800	2.165681	2.0800	2.090800	0.085681	0.0000	0.010800
0.65	2.2675	2.376435	2.2675	2.281231	0.108935	0.0000	0.013731
0.70	2.4700	2.606058	2.4700	2.487150	0.136058	0.0000	0.017150
0.75	2.6875	2.854845	2.6875	2.708594	0.167345	0.0000	0.021094
0.8	2.9200	3.123095	2.9200	2.945600	0.203095	0.0000	0.025600
0.85	3.1675	3.411105	3.1675	3.198206	0.243605	0.0000	0.030706
0.90	3.4300	3.719172	3.4300	3.466450	0.289172	0.0000	0.036450
0.95	3.7075	4.047595	3.7075	3.750369	0.340095	0.0000	0.042869
1.00	4.0000	4.396670	4.0000	4.050000	0.396670	0.0000	0.050000

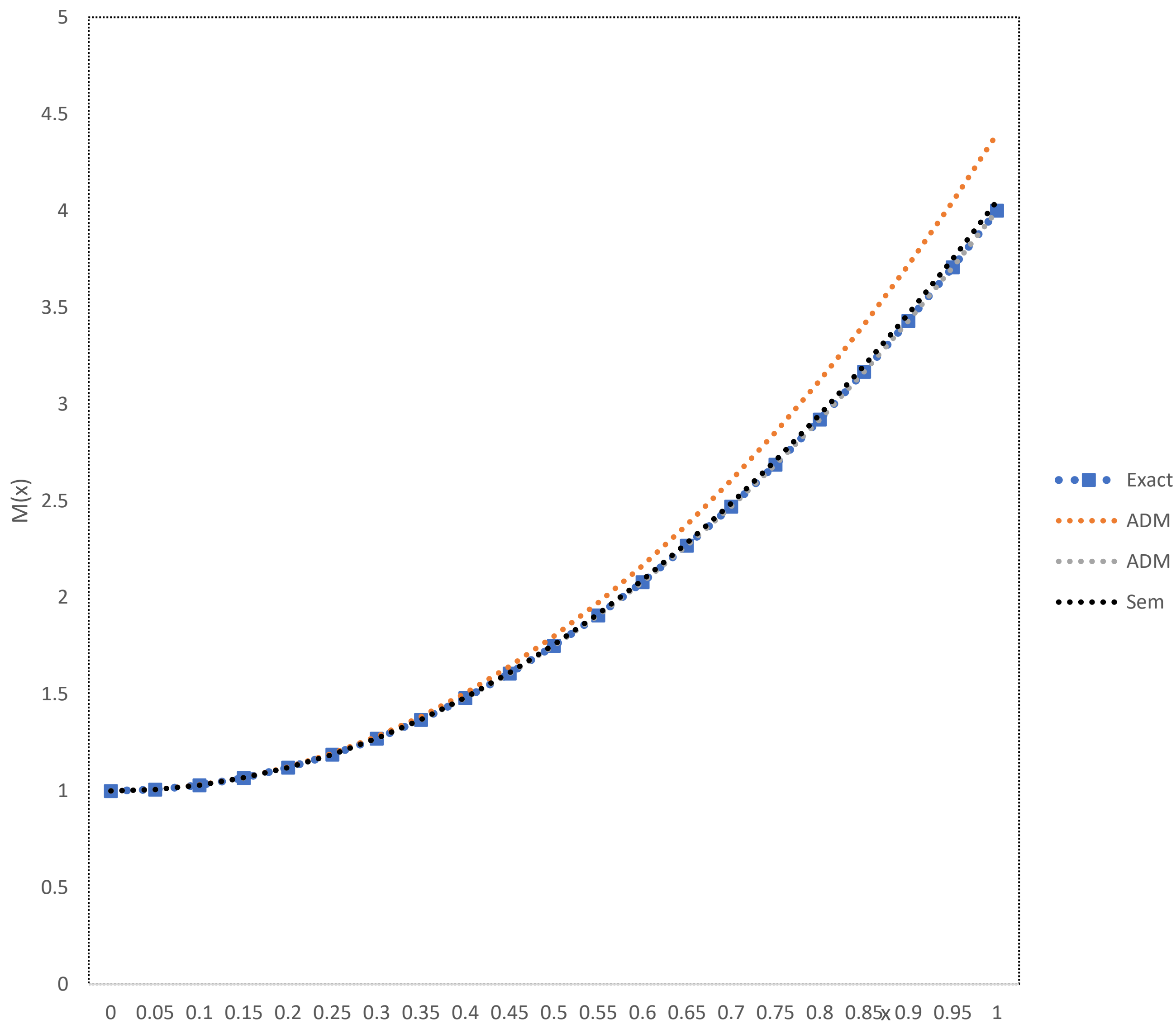

Figure 3.7: Exact and Approximate Solution by ADM, MADM and SEM for Example 2(M(x)) with step size 0.05

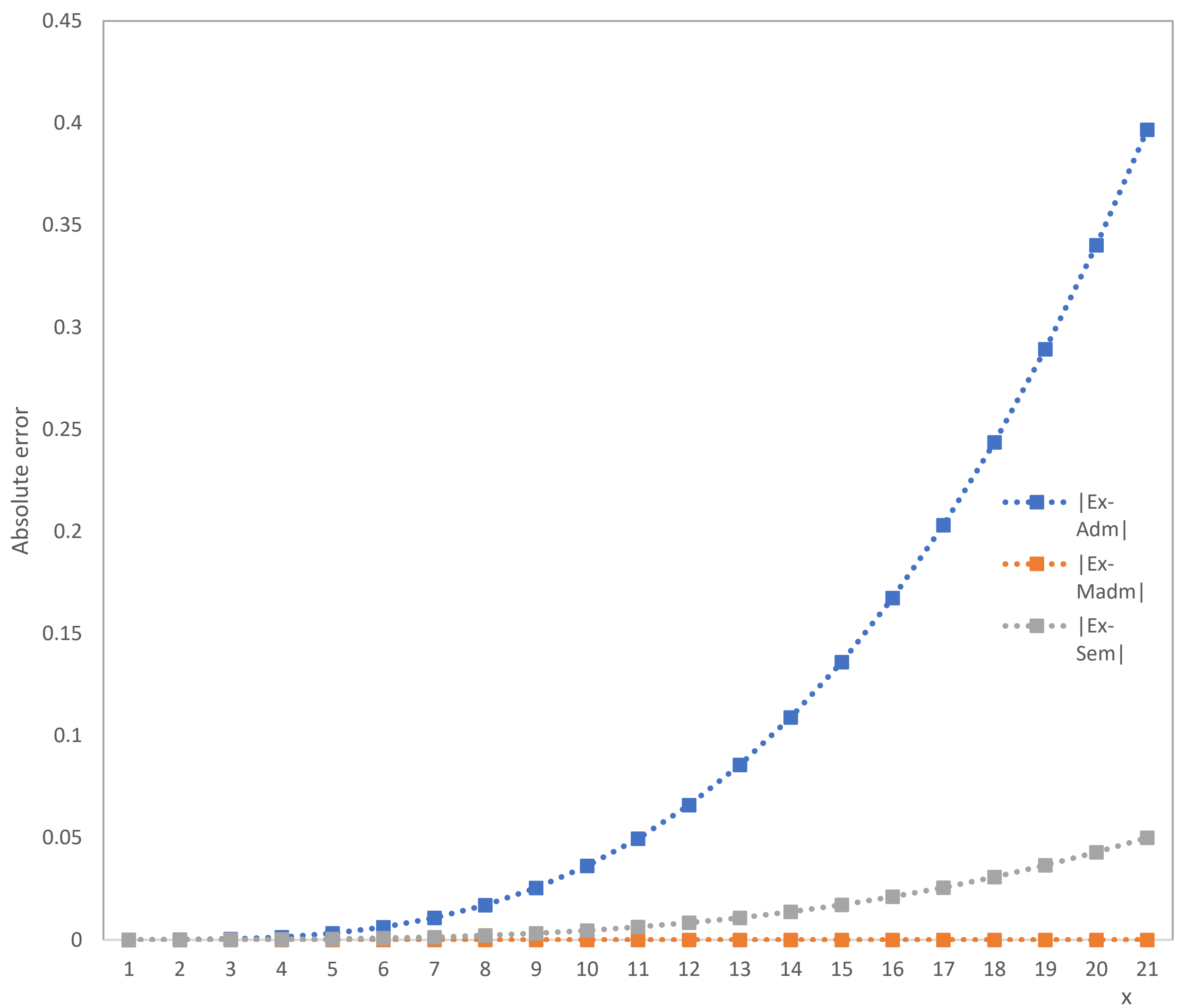

Figure 3.8: Absolute error of ADM, MADM and SEM for Example 2(M(x)) with step size 0.05

Table 3.8. Root Mean Square Error Between the Exact and Approximate Solution by ADM, MADM and SEM for Example 2 M(x) with step size 0.05

Root Mean Square Error (RMSE)	
ADM	0.159189
MADM	0
SEM	0.020066

Table 3.9: Exact and Approximate Solution by ADM, MADM and SEM for example 2(N(X)) with step size 0.01

x	Exact	ADM	MADM	SEM	|Ex-Adm|	Ex-Mad||	| Ex-Sem|
0.00	-1.000000	-1.000000	-1.000000	-1.0000000	0.00000000	0.0000000	0.0000000
0.01	-0.979999	-0.979894	-0.979999	-0.9799585	0.00010475	0.0000000	4.05E-05
0.02	-0.959992	-0.959569	-0.959992	-0.9598280	0.00042303	0.0000000	0.0001640
0.03	-0.939973	-0.939012	-0.939973	-0.9395995	0.00096086	0.0000000	0.0003735
0.04	-0.919936	-0.918212	-0.919936	-0.9192640	0.00172427	0.0000000	0.0006720
0.05	-0.899875	-0.897156	-0.899875	-0.8988125	0.00271927	0.0000000	0.0010625
0.06	-0.879784	-0.875832	-0.879784	-0.8782360	0.00395191	0.0000000	0.0015480
0.07	-0.859657	-0.854229	-0.859657	-0.8575255	0.00542819	0.0000000	0.0021315
0.08	-0.839488	-0.832334	-0.839488	-0.836672	0.00715415	0.0000000	0.0028160
0.09	-0.819271	-0.810135	-0.819271	-0.8156665	0.00913581	0.0000000	0.0036045
0.10	-0.799000	-0.787621	-0.799000	-0.7945000	0.01137920	0.0000000	0.0045000
0.11	-0.778669	-0.764779	-0.778669	-0.7731635	0.01389034	0.0000000	0.0055055
0.12	-0.758272	-0.741597	-0.758272	-0.7516480	0.01667526	0.0000000	0.0066240
0.13	-0.737803	-0.718063	-0.737803	-0.7299445	0.01973998	0.0000000	0.0078585

0.14	-0.717256	-0.694165	-0.717256	-0.7080440	0.02309052	0.0000000	0.0092120
0.15	-0.696625	-0.669892	-0.696625	-0.6859375	0.02673293	0.0000000	0.0106875
0.16	-0.675904	-0.645231	-0.675904	-0.6636160	0.0306732	0.0000000	0.0122880
0.17	-0.655087	-0.620170	-0.655087	-0.6410705	0.03491738	0.0000000	0.0140165
0.18	-0.634168	-0.594697	-0.634168	-0.6182920	0.03947149	0.0000000	0.0158760
0.19	-0.613141	-0.568799	-0.613141	-0.5952715	0.04434156	0.0000000	0.0178695
0.20	-0.592000	-0.542466	-0.592000	-0.5720000	0.0495336	0.0000000	0.0200000

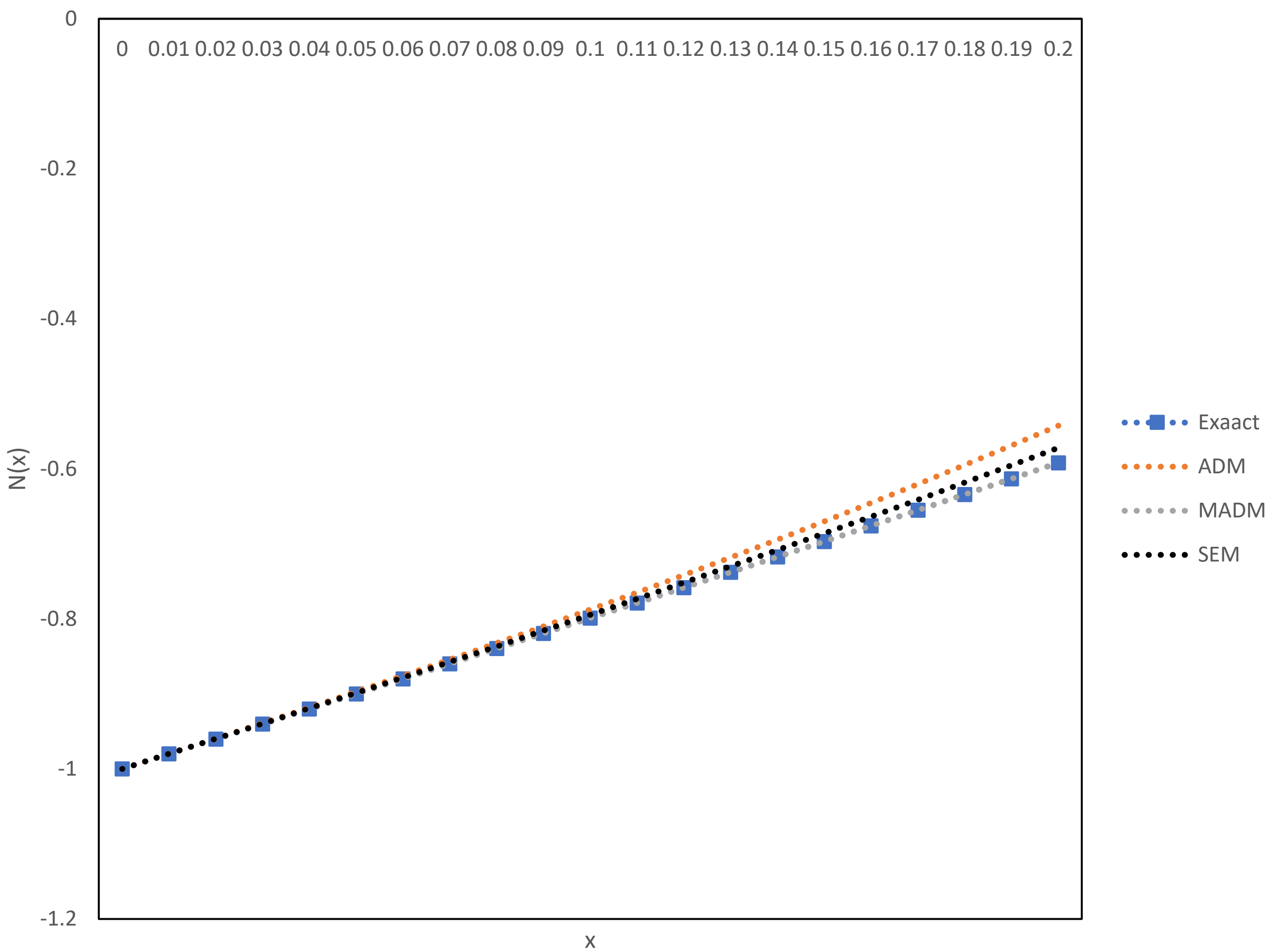

Figure 3.9: Exact and Approximate Solution by ADM, MADM And SEM For Example 2 N(X) with step size 0.01

Table 3.10. Root Mean Square Error Between the Exact and Approximate Solution by ADM, MADM and SEM for Example 2 N(x) with step size 0.01

Root Mean Square Error (RMSE)	
ADM	0.022432
MADM	0
SEM	0.009007

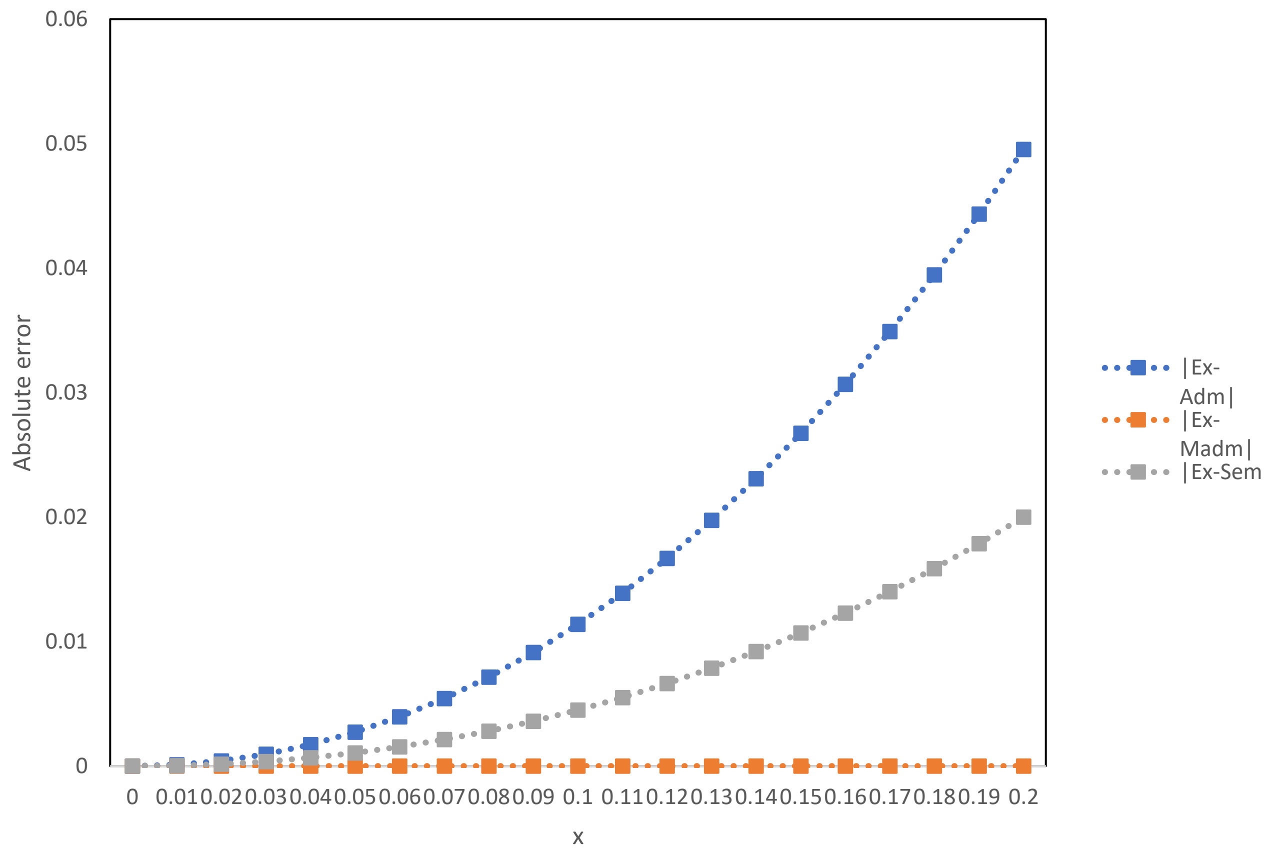

Figure 3.10. Absolute Difference Between the Exact and Approximate Solution by ADM, MADM and SEM for Example 2 N(x) with step size 0.01

Table 3.11: Exact and Approximate Solution by ADM, MADM And SEM For Example 2 N(X) with step size 0.05

x	Exact	ADM	MADM	SEM	|Ex-Adm|	|Ex-Mad|	|Ex-Sem|
0.00	-1.000000	-1.0000000	-1.000000	-1.0000000	0.000000	0.0000000	0.0000000
0.05	-0.899875	-0.8971557	-0.899875	-0.8988125	0.002719	0.0000000	0.0010625
0.10	-0.799000	-0.7876208	-0.799000	-0.7945000	0.011379	0.0000000	0.0045000
0.15	-0.696625	-0.6698921	-0.696625	-0.6859375	0.026733	0.0000000	0.0106875
0.20	-0.592000	-0.5424664	-0.592000	-0.5720000	0.049534	0.0000000	0.0200000
0.25	-0.484375	-0.4038406	-0.484375	-0.4515625	0.080534	0.0000000	0.0328125
0.30	-0.373000	-0.2525116	-0.373000	-0.3235000	0.120488	0.0000000	0.0495000
0.35	-0.257125	-0.0869762	-0.257125	-0.1866875	0.170149	0.0000000	0.0704375
0.40	-0.136000	0.0942688	-0.136000	-0.0400000	0.230269	0.0000000	0.0960000
0.45	-0.008875	0.29272648	-0.008875	0.1176875	0.301601	0.0000000	0.1265625
0.50	0.125000	0.50990000	0.125000	0.2875000	0.384900	0.0000000	0.1625000
0.55	0.266375	0.74729253	0.266375	0.4705625	0.480918	0.0000000	0.2041875
0.60	0.416000	1.00640720	0.416000	0.6680000	0.590407	0.0000000	0.2520000
0.65	0.574625	1.28874718	0.574625	0.8809375	0.714122	0.0000000	0.3063125
0.70	0.743000	1.59581560	0.743000	1.1105000	0.852816	0.0000000	0.3675000
0.75	0.921875	1.92911563	0.921875	1.3578125	1.007241	0.0000000	0.4359375
0.80	1.112000	2.29015040	1.112000	1.6240000	1.178150	0.0000000	0.5120000
0.85	1.314125	2.68042308	1.314125	1.9101875	1.366298	0.0000000	0.5960625
0.90	1.529000	3.10143680	1.529000	2.2175000	1.572437	0.0000000	0.6885000
0.95	1.757375	3.55469473	1.757375	2.5470625	1.797320	0.0000000	0.7896875
1.00	2.000000	4.04170000	2.000000	2.9000000	2.041700	0.0000000	0.9000000

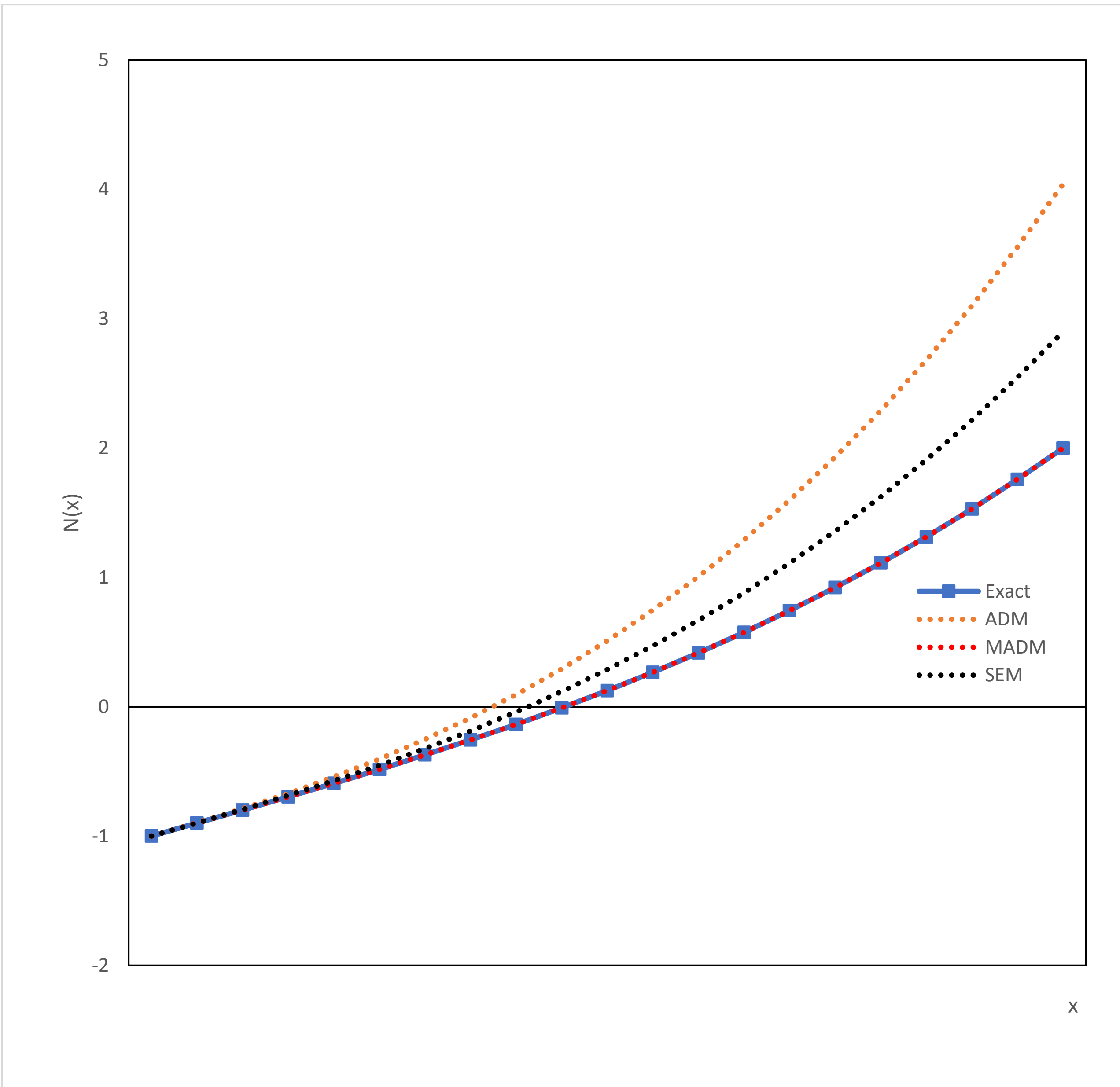

Figure 3.11: Exact and Approximate Solution by ADM, MADM And SEM For Example 2 N(X) with step size 0.05

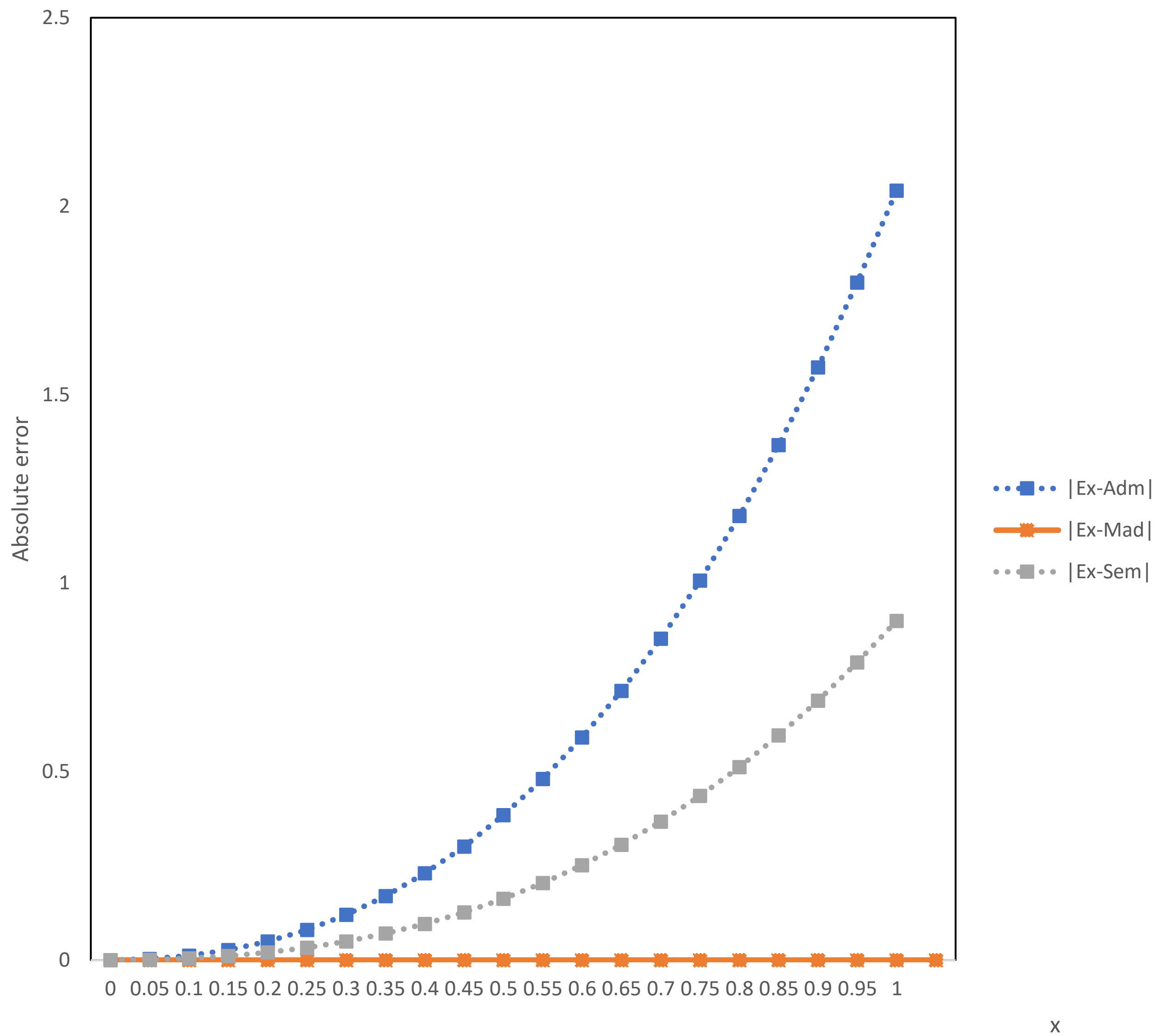

Figure 3.12: Absolute error ADM, MADM And SEM For Example 2 N(X) with step size 0.05

Table 3.12. Root Mean Square Error Between the Exact and Approximate Solution by ADM, MADM and SEM for Example 2 N(x) with step size 0.05

Root Mean Square Error (RMSE)	
ADM	0.881106
MADM	0
SEM	0.384823

4. Discussion

In this comparative analysis, we evaluated the solutions to Fredholm linear integro-differential equations using three different methods: The Adomian Decomposition Method (ADM), the Modified Adomian Decomposition Method (MADM), and the Series Expansion Method (SEM). The effectiveness of each method was assessed based on numerical accuracy, convergence rate, computational efficiency, and ease of implementation.

The ADM is a powerful tool for solving linear integro-differential equations, providing solutions in the form of a rapidly convergent series. This method decomposes the original problem into a series of simpler sub-problems, which are easier to solve. It ADM provides highly accurate solutions for Fredholm integro-differential equations, especially for problems with smooth and well-behaved kernels. The series solution obtained by ADM converges rapidly, reducing computational overhead. The implementation of ADM is straightforward, involving recursive computation of Adomian polynomials, which simplifies the process.

The MADM typically yields more accurate results compared to ADM. The convergence rate of MADM is generally faster than that of ADM, leading to more efficient computations. While slightly more complex than ADM, MADM remains relatively easy to implement and offers significant improvements in terms of precision and efficiency.

The Series Expansion Method involves expressing the solution as a series and determining the coefficients through various techniques, such as power series or Fourier series. The convergence is slower compared to ADM and MADM. SEM is more challenging to implement, requiring careful consideration of the series type and the computation of coefficients.

Comparing the methods used, it's evident from the results in the examples provided. The techniques exhibit strength, effectiveness, and offer more precise approximations, often yielding closed-form solutions where possible. When the methods were applied to linear Fredholm integro-differential equations with separable kernels, the outcomes from different approaches tend to be quite similar. In comparison to SEM, both MADM and ADM are notably simpler particularly, the MADM solution stands out for its superior precision and reduced computational requirements compared to ADM and SEM. Moreover, MADM shows quicker convergence than ADM and SEM, requiring fewer computations.

Tables 3.1- 3.12 illustrate the comparison of result, absolute error and root mean square(RMSE) among ADM, MADM, and SEM concerning exact solutions. The error analyses depicted in these tables confirm that MADM consistently outperforms ADM and SEM. Additionally, statistical assessments underscore that MADM achieves higher precision more rapidly than ADM and SEM. Furthermore, when the step size was increased from 0.01 to 0.05, ADM deviated more from the Exact solution while SEM tends more to the Exact solution. The comparison of results, absolute errors and RMSE indicates SEM's superiority over ADM.

The visual representations in Figures 3.1 to 3.12 complement these findings, offering a graphical overview of the analysis.

5. Conclusion

The Modified Adomian Decomposition Method stands out as the most effective approach for solving Fredholm linear integro-differential equations, offering superior accuracy, convergence, and computational efficiency. The Adomian Decomposition Method also performs well, providing a balance of accuracy and ease of implementation. The Series Expansion Method, while powerful, is best suited for problems where the series convergence is well-understood and can be effectively managed.

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